Circles

Review Exercise Answers

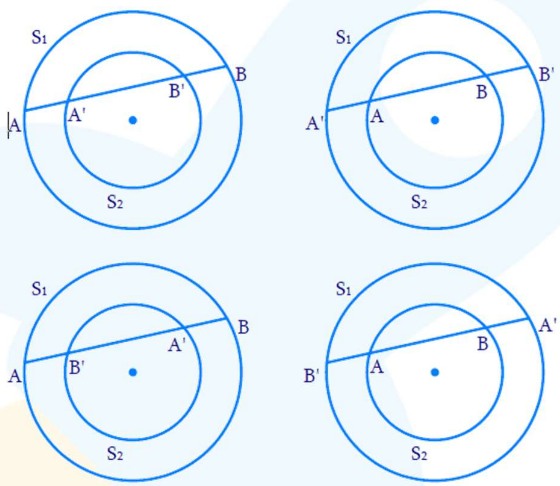
Level – 2

Multiple Options may be Correct

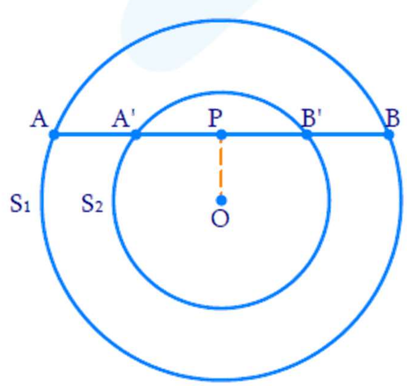
S1. (A), (B), (C) and (D). (A) is correct as follows for a cyclic parallelogram ABCD, $\angle A$ and $\angle C$ will be equal as well as supplementary, which means that both are equal to 90º. Thus, ABCD must be a rectangle. Proving the correctness of the other statements is left to you as an exercise.

Miscellaneous

S2. Observe that a number of possibilities exist. The radius of S_1 might be larger than S_2, or vice-versa. A' may be on the same or opposite side of A as B, etc. These possibilities are shown below:



However, in each case, $AA' = BB'$ will hold true. We will prove this for the first configuration, but the proofs for the other cases will be smaller.
Consider the following figure. O is the common center of S_1 and S_2, and $OP \perp AB$:



For S_2, OP is perpendicular to its chord $A'B'$, and so:

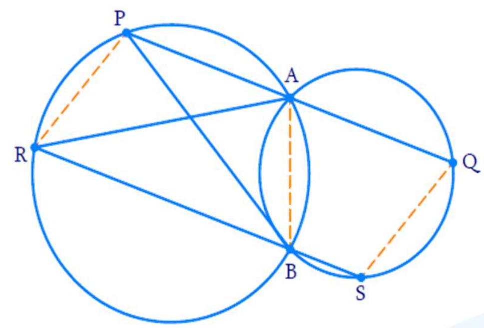
$$A'P = B'P$$

Similarly, for S_1, OP is perpendicular to its chord AB, so that:

Thus,

$$AP - A'P = BP - B'P$$
$$\Rightarrow AA' = BB'$$

S3. Consider the following figure:



Our effort will be directed towards proving that PQRS is a parallelogram, from which it will follow that PQ = RS. Compare ΔPRB with ΔARB:

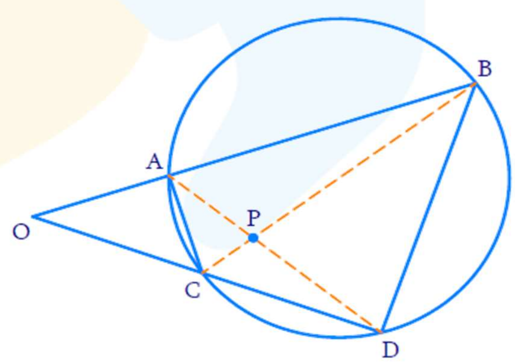
* RB = RB (common)
* $\angle RPB = \angle RAB$ (angles subtended by RB)
* $\angle RBR = \angle ARB$. This is because $\angle PBR = \angle BPA$ (alternate angles), and $\angle BPA = \angle ARB$ because these are angles subtended by the same chord AB.

Thus, $\angle PRB \cong \angle ARB$ by the ASA criterion. This mean. This means that PR = AB, and $\angle PRB = \angle ABR$.

Similarly, we can prove that QS = AB and $\angle QSB = \angle ABS$. Also, since $\angle ABR + \angle ABS$ = 180º , we have $\angle PRB + \angle QSR$ =180º, from which we conclude that PR||QS.

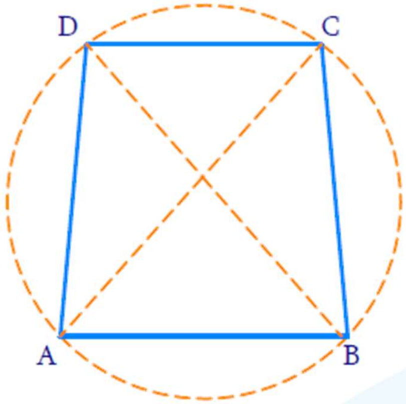
It is now clear that PQRS is a parallelogram. Thus, PQ = RS.

S4. Consider the following figurer:

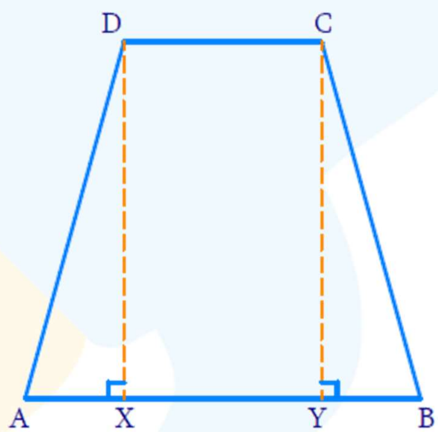


We have to determine whether P can be the center of the circle. If P **is** the center of the circle, then AD and BC would be diameters of the circle bisecting each other at P, which would further mean that ABDC would have been a parallelogram, with AB || CD. However, that is clearly not true, because AB and CD (when extended) meet at O. Thus, P cannot be the center of the circle.

S5. An isosceles trapezium is one in which the non-parallel sides are equal. Suppose that ABCD is an isosceles trapezium with AB || CD and AD = BC. Draw the circle passing through A, B and C. We will prove that it also passes through D:



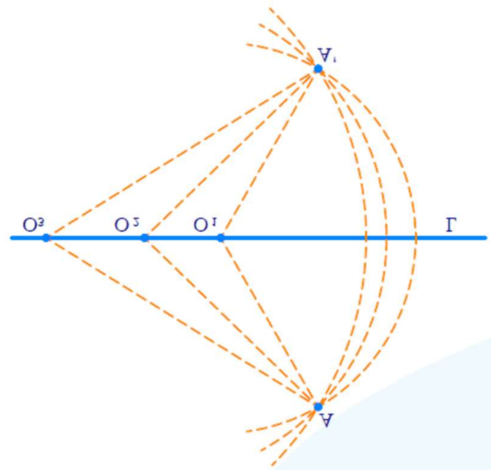
Compare ΔADB with ΔACB:
* AD = BC (the trapezium is isosceles)
* AB = AB (common)
* $\angle DAB = \angle CBA$. How? To prove that, drop perpendiculars DX and CY onto AB:



Since DX and CY are perpendiculars between the same parallels, they are equal. Thus, $\Delta DAX \cong \Delta CBY$ by the RHS criterion, which means that $\angle A \left(\text{or } \angle DAB\right)$ is equal to $\angle B\left(\text{or } \angle CBA\right)$.

Coming back, we see that $\Delta ADB \cong \Delta ACB$ by the SAS criterion, so that $\angle ADB = \angle ACB$. This means that the angle which AB subtends at D is the same as the angle which it subtends at C. Clearly, D must also lie of the circle passing through A, B and C.

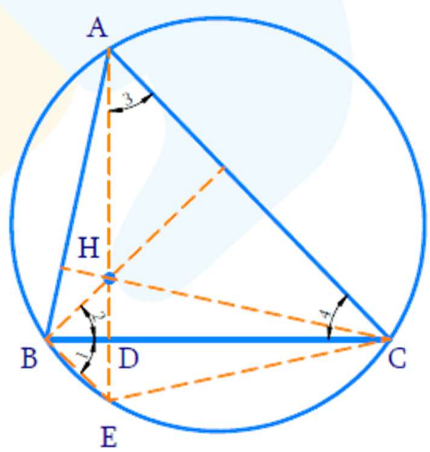
S6. This problem might seem difficult at first glance, but its solution is extremely simply. In the following figure, A' is the mirror reflection of A in the line L. Note that since A and L are fixed, A' is also fixed:



We have also shown segments of three different example circles, each of whose center lies on L. Clearly, every such circle, because it passes through A, must also pass through A'. For example, since $O_1A = O_1A'$ (as L is the perpendicular bisector of AA'), A' must also lie on the circle whose center is O_1 and which passes through A. This is true for every circle of the family.

Thus, every circle of the family also passes through A'.

S7. Consider the following figure:

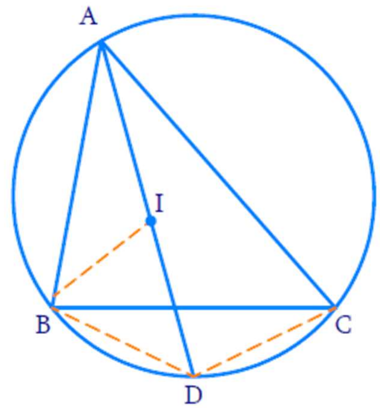


Compare ΔBED with ΔBHD:

* $\angle 1 = \angle 2$ (from the observation in the figure)
* BD = BD (common)
* $\angle BDE = \angle BDH = 90^o$

Thus, $\Delta BED \cong \Delta BHD$ by the ASA criterion, so that DE = HD.

S8. Recall that the incenter of a triangle is the point of concurrence of its angle bisectors. Consider the following figure:



Note that since AD is the angle bisector of $\angle A$, we have $\angle BAD = \angle CAD,$ and thus, $\overset{\frown}{BD} = \overset{\frown}{CD}$, which means that BD = CD or DB = DC. Now, we will prove that DI is also equal to DB and DC. For that, we will calculate the magnitudes of $\angle DBI$ and $\angle DIB$ in $\angle DBI$.

* $\angle DBI = \angle DBC + \angle CBI$

$$= \angle DAC + \frac{1}{2}\angle B \ (\text{how ?})$$

$$= \frac{1}{2}\angle A + \frac{1}{2}\angle B = \frac{1}{2}(\angle A + \angle B)$$

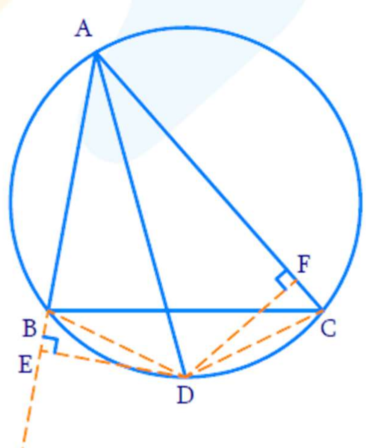
* $\angle DIB = \angle DAB + \angle IBA$ (exterior angle theorem)

$$= \angle\angle + \frac{1}{2}\angle B$$

$$= \frac{1}{2}(\angle A + \angle B)$$

Clearly, $\angle DBI = \angle DIB,$ which implies that DB = DI. Thus, DB = DC = DI.

S9. Consider the following figure:



Since AD is the angle bisector of $\angle A, \angle BAD = \angle CAD,$ which means that $\overset{\frown}{BD} = \overset{\frown}{CD}$, and thus BD = CD. Also, since D lies on the angle bisector of $\angle A$, it must be equidistant from AB and AC, i.e., DE = DF. Thus, $\Delta BDE \cong \Delta CDF$ by the RHS criterion, so that BE = CF.

Now,

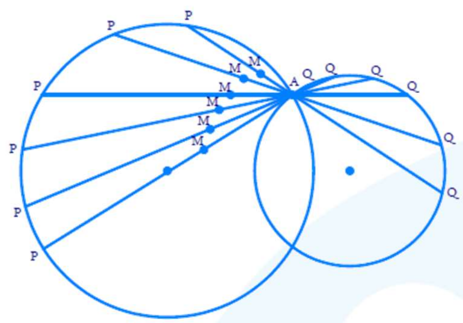
AB = AE – BE ... (1)

AC = AF + CF ... (2)

Also, AE = AF (why). Adding (1) and (2), AB + AC = AE + AF = 2AE

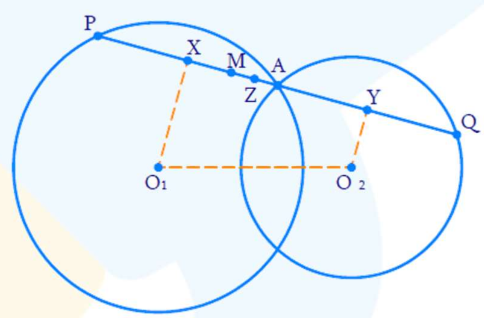
$$\Rightarrow \mathrm{AE} = \frac{\mathrm{AB} + \mathrm{AC}}{2}$$

S10. Observe the following figure carefully. The two circles intersect at A and B. Through A, a number of lines have been drawn. The intersection of the lines with the circles are the (variable) points P and Q. M (represented by the large dot) is the mid-point of PQ. Note how the position of M changes as the inclination of PQ changes.



We have to determine the path traced out by M as the inclination of PQ changes.

In the following figure, we have drawn just one instance of PQ. X and Y are the mid-points of AP and AQ respectively. Note that $\mathrm{O_1X} \perp \mathrm{PAQ}$ and $\mathrm{O_1Y} \perp \mathrm{PAQ}$ (O_1 and O_2 are the two centers). Z is the mid-point of AM.



Note that MP = MQ, which means that

$$\mathrm{AP} - \mathrm{AM} = \mathrm{AQ} + \mathrm{AM} \Rightarrow \mathrm{AM} = \frac{\mathrm{AP} - \mathrm{AQ}}{2}$$

Now, let us evaluate ZX and ZY:

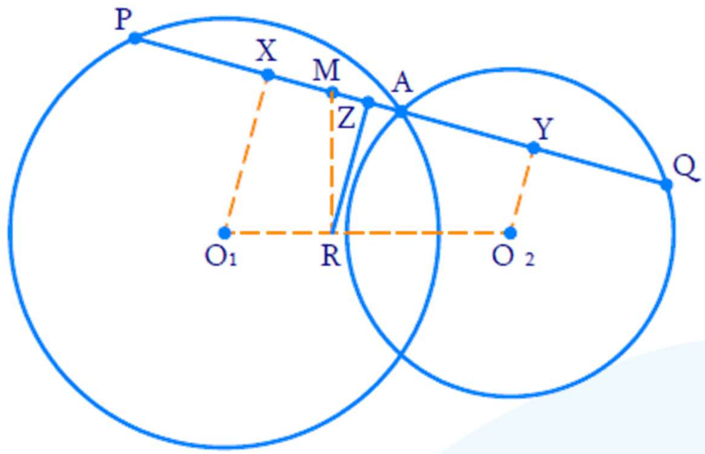
$$\mathrm{ZX} = \mathrm{AX} - \mathrm{AZ} = \frac{\mathrm{AP}}{2} - \frac{\mathrm{AM}}{2}$$

$$= \frac{\mathrm{AP}}{2} - \left(\frac{\mathrm{AP} - \mathrm{AQ}}{4}\right) = \frac{\mathrm{AP} + \mathrm{AQ}}{4}$$

$$\mathrm{ZY} = \mathrm{AY} + \mathrm{AZ} = \frac{\mathrm{AQ}}{2} + \frac{\mathrm{AM}}{2}$$

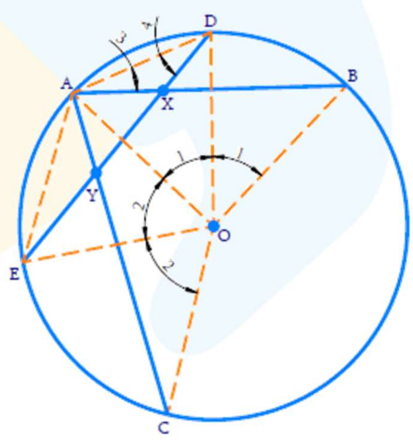
$$= \frac{\mathrm{AQ}}{2} + \left(\frac{\mathrm{AP} - \mathrm{AQ}}{4}\right) = \frac{\mathrm{AP} + \mathrm{AQ}}{4}$$

Thus, ZX = ZY, which means that Z is the mid-point of XY. Through Z, draw $RZ \perp PAQ$ meeting O_1O_2 in R:



Observe that R must be the mid-point of O_1O_2 (why?), and hence R is fixed. Also, note that since RZ is the perpendicular bisector of AM, ARM = RA. Thus, the distance of the variable point M from the fixed point R is constant, equal to RA. This means that M traces out a circle, whose center R is the mid-point of O_1O_2, and whose radius is equal to RA (or RB).

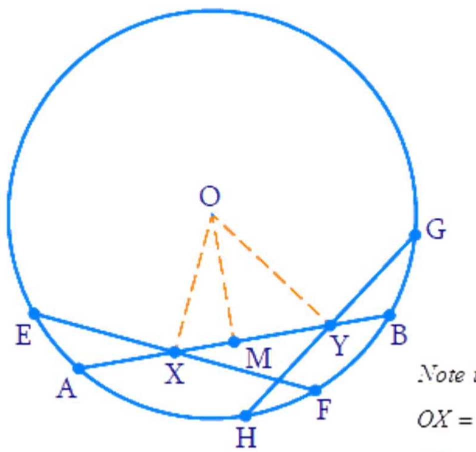
S11. Observe the following figure, and note that D and E are mid-points (respectively) of minor $\overparen{AB}$ and minor $\overparen{AC}$. Since D is the mid-point of $\overparen{AB}, \angle DOB = \angle DOA$.
Similarly, $\angle EOC = \angle EOA$:



We have shown that DE is inclined to AB at an angle of $\angle DXB = \frac{1}{2}(\angle 1 + \angle 2)$. Similarly, we can show that DE is inclined to AC at an angle of $\angle EYC = \frac{1}{2}(\angle 1 + \angle 2)$.

Thus, DE is equal inclined to AB and AC.

S12. In the following figure, EF and GH are two chords which get bisected by AB at X and Y respectively. M is the mid-point of AB, and MX < MY:



Note that

$OX = \sqrt{OM^2 + MX^2}$

$OY = \sqrt{OM^2 + MY^2}$

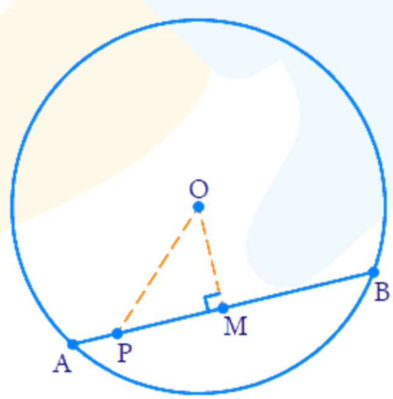
$\Rightarrow OX < OY \ (\because MX < MY)$

Also, $OX \perp EY$ *and* $OY \perp GH,$

since x and Y are the mid-point of EF and GH respectively

We conclude that since OX < OY, EF is near to O then GH, and thus EF must be greater than GH.

S13. Consider the following figure. AB is a variable chord which passes through the (fixed) point P. M is the (variable) mid-point

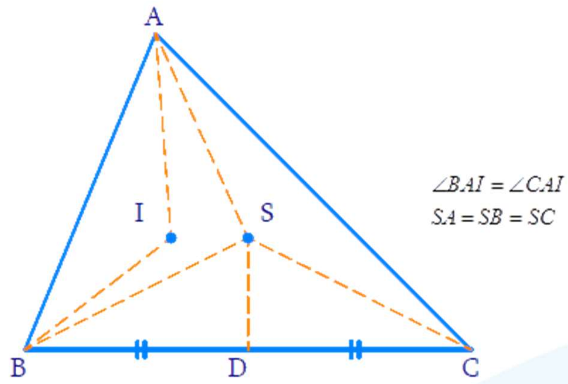


Since M is the mid-point of AB

$OM \perp AB$

As AB varies (all the while passing through P), the position of M will vary. However, OM will always remain perpendicular to AB, that is, the (fixed) segment OP will always subtend and angle of 90º at M.
Thus, M will trace out a circle with OP as its diameter.

S14. Consider the following figure (in this particular scenario, we have assumed $\angle B > \angle C$):



We have:

$\angle BAS = \angle BAI + \angle IAS$ ... (1)

$\angle CAS = \angle CAI - \angle IAS$ ... (2)

By (1) – (2), we have:

$$2\angle IAS = \angle BAS - \angle CAS \qquad (\text{how ?})$$

$$= \angle ABS - \angle ACS \qquad (\text{how ?})$$

$$= \left(\angle ABS + \underbrace{\angle SBD}_{\text{new term}}\right) - \left(\angle ACS + \underbrace{\angle SCD}_{\text{new term}}\right) \qquad \{\angle SBD = \angle SCD\}$$

$$= \angle B - \angle C$$

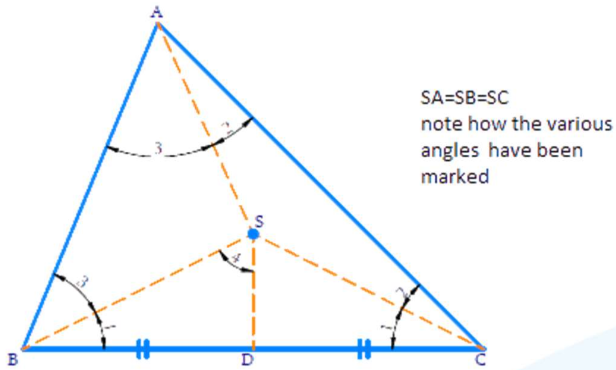
$$\Rightarrow \quad \angle IAS = \frac{1}{2}(\angle B - \angle C)$$

In this case, we took $\angle B > \angle C$. If $\angle B$ was smaller, we would have obtained:

$$\angle IAS = \frac{1}{2}(\angle C - \angle B)$$

Thus, $\angle IAS$ is equal to half of the difference between $\angle B$ and $\angle C$. If $\angle B = \angle C$, $\angle IAS$ will be zero.

S15. The circumcenter is the point of concurrence of the perpendicular bisectors of the sides of the triangle, and is equidistant from each vertex. Consider the following figure:



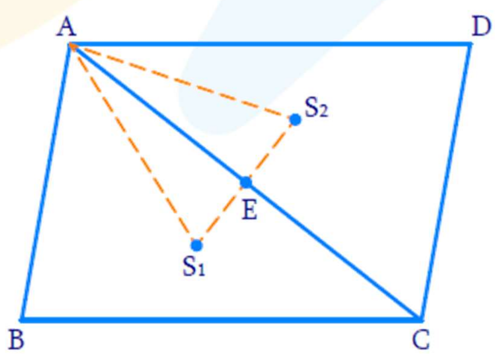
We have:

$$\angle A + \angle B + \angle C = 180^o$$
$$\Rightarrow (\angle 2 + \angle 3) + (\angle 3 + \angle 1) + (\angle 1 + \angle 2) = 180^o$$
$$\Rightarrow \angle 1 + \angle 2 + \angle 3 = 90^o$$
$$\Rightarrow \angle 1 + \angle A = 90^o$$
$$\Rightarrow \angle A = 90^o - \angle 1$$

Also, in $\Delta BSD,$ we see that $\angle 4 = 90^o - \angle 1$. Thus,

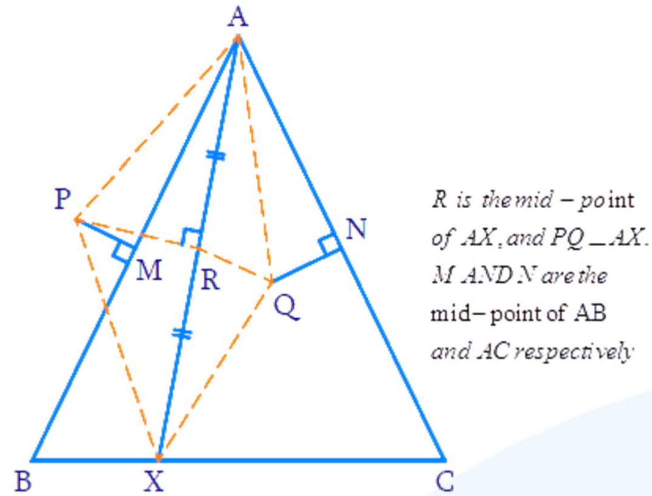
$$\angle A = \angle BAC = \angle 4 = \angle BSD$$

S16. Consider the following figure, which shows $S_1E = S_2E$. Note that E is the mid-point of AC. We have joined AS_1 and AS_2:

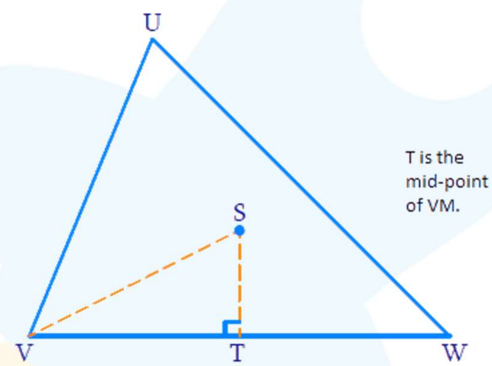


Now, since $\Delta ACB \cong \Delta ACD$, the circumcircles of the two triangles will have the same radius. Observe that AS_1 is a rardius for the circumcircle of ΔACB, while AS_2 is radius for the circumcirle of ΔACD. Thus, $AS_1 = AS_2$. This means that $\Delta AS_1E \cong \Delta AS_2E$ by the RHS criterion, so that $S_1E = S_2E$.

S17. Consider the following figure:



We will make use of the following result. Consider a ΔUVW with S as its circumcentre:



Then $\angle VST = \angle U$. We have encountered this result earlier, in [ref]. Coming back to our original problem, we note that $\angle APM = \angle C$, and $\angle AQN = \angle B$. But $\angle B = \angle C$, so that

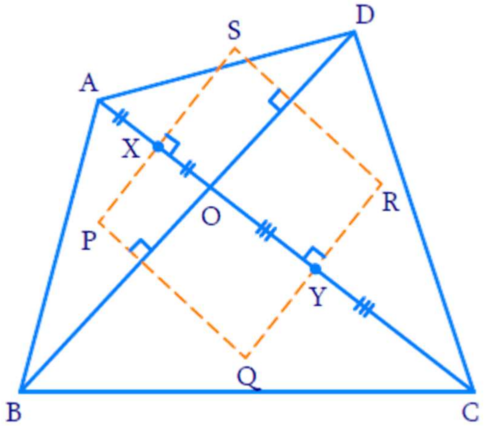
$$\angle APM = \angle AQN$$

Also, since AM = AN = $\frac{1}{2}AB$ or $\frac{1}{2}AC$, ΔAPM must be congruent to ΔAQN by ASA criterion. This means that AP = AQ.

Finally, we see that because of this, $\Delta APR \cong \Delta AQR$ by the RHS criterion, so PR = QR.

We see that the diagonals of APXQ bisect each other, so it must be a parallelogram. In fact, if is a rhombus, since the diagonals are perpendicular to each other.

S18. Consider the following figure:



P, Q, R and S are the four circumcenters. Note that X, P and S are collinear, because $XP \perp OA$ and $XS \perp OA$. Thus, we can say that $PS \perp OA$, or $PS \perp AC$. Similarly, $QR \perp AC$. Since both PS and QR are perpendicular to the same line, we conclude that PS || QR. In a similar manner, we can show that PQ || SR. Hence, PQRS is a parallelogram.